

Aero First-Principles Notebook

Heat-Exchanger Tube: Pressure and Thermal FEA Verification

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Scope. Synthetic educational example. These derivations and the companion small finite-element calculations verify an idealized model, not a manufactured exchanger. No new HX-03, HX-04, or HX-CFD qualification is claimed. This is not experimental validation. The public notebook does not expose the private Aero corpus, local GPU, or arbitrary compilation.

1 Frozen assumed inputs

Quantity	Symbol	Value
Inner / outer radius	a, b	10, 11 mm
Length	L	1 m
Internal / external gauge pressure	p_i, p_o	2, 0 MPa
Elastic modulus / Poisson ratio	E, ν	190 GPa, 0.30
Expansion coefficient	α	$16 \times 10^{-6} \text{ K}^{-1}$
Uniform temperature rise	ΔT	80 K

These are assumed elastic properties, not a certified material specification. Pressure-only and thermal-only tests are separate. Do not add the answers without re-closing the combined load and restraint problem.

2 Pressure-only closed-end tube

Take tensile stress as positive. For a straight long tube away from the end caps, rotational symmetry and radial equilibrium give

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0. \quad (1)$$

For constant axial strain, kinematics give $\epsilon_r = u'_r$ and $\epsilon_\theta = u_r/r$. Isotropic Hooke's law and equilibrium then reduce to

$$u''_r + \frac{u'_r}{r} - \frac{u_r}{r^2} = 0, \quad u_r = C_1 r + \frac{C_2}{r}. \quad (2)$$

Substitution into the constitutive law yields the Lamé form

$$\sigma_r = A - \frac{B}{r^2}, \quad \sigma_\theta = A + \frac{B}{r^2}. \quad (3)$$

Apply $\sigma_r(a) = -p_i$ and $\sigma_r(b) = -p_o$:

$$A = \frac{p_i a^2 - p_o b^2}{b^2 - a^2}, \quad B = \frac{a^2 b^2 (p_i - p_o)}{b^2 - a^2}. \quad (4)$$

The closed-end axial force balance is

$$\sigma_z \pi (b^2 - a^2) = \pi (p_i a^2 - p_o b^2), \quad \sigma_z = A. \quad (5)$$

This is *not* the open-ended condition $\sigma_z = 0$, and not plane strain $\epsilon_z = 0$. Hooke's law gives

$$u_r(r) = \frac{(1 - 2\nu)Ar + (1 + \nu)B/r}{E}, \quad \epsilon_z = \frac{(1 - 2\nu)A}{E}. \quad (6)$$

At the bore, $\sigma_r = -2.000$ MPa, $\sigma_\theta = 21.047619$ MPa and $\sigma_z = 9.523810$ MPa. The outer displacement is approximately $0.937343 \mu\text{m}$. The thin-wall estimate $p_i a / (b - a) = 20$ MPa is lower; it is not substituted for the thick-wall reference.

Equation checks. Substitute the two stress expressions into equilibrium: $2B/r^3 - 2B/r^3 = 0$. Boundary tractions and end force balance are separately checked by executable tests. Units are $[A] = \text{Pa}$, $[B] = \text{Pa m}^2$ and $[u] = \text{m}$.

3 Independent radial finite-element calculation

The companion script assembles linear radial elements and one constant axial strain degree of freedom. For each element,

$$\epsilon = \mathbf{B}\mathbf{q}, \quad \mathbf{B} = \begin{bmatrix} N'_1 & N'_2 & 0 \\ N_1/r & N_2/r & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{D} = 2\mu\mathbf{I} + \lambda\mathbf{1}\mathbf{1}^T. \quad (7)$$

Here $\mu = E/[2(1 + \nu)]$ and $\lambda = E\nu/[(1 + \nu)(1 - 2\nu)]$. The weak form is

$$\mathbf{K} = \sum_e \int_e \mathbf{B}^T \mathbf{D} \mathbf{B} 2\pi r L dr, \quad \mathbf{K}\mathbf{q} = \mathbf{f}. \quad (8)$$

The inner radial load is $2\pi a L p_i$, outer load $-2\pi b L p_o$ and the generalized axial load is $L\pi(p_i a^2 - p_o b^2)$. Three-point Gauss integration is used. Refine through 4, 8, 16, 32 and 64 radial elements; compare displacement and volume-weighted Gauss-point stress error with the independently derived Lamé reference. Exact numerical values are retained in `results.json` and `pressure-convergence.csv`. The declared finest-mesh checks are: outer-displacement error below 0.1%, stress relative L_2 error below 2%, and normalized equilibrium residual below 10^{-9} . These are educational verification checks, not Aero's production mesh-independence criteria.

4 Thermal-only tube: two different restraint problems

Treat the tube as a uniform axial bar with area $S = \pi(b^2 - a^2)$ and traction-free transverse surfaces. Its constitutive equation is

$$\sigma_z = E \left(\frac{du}{dx} - \alpha \Delta T \right). \quad (9)$$

With one axial datum and the other end free, $\sigma_z = 0$, so

$$u(L) = \alpha \Delta T L = 1.280 \text{ mm}. \quad (10)$$

With *axial* displacement fixed at both ends, $\int_0^L u' dx = 0$. Uniform temperature and uniform area therefore give

$$\sigma_z = -E\alpha\Delta T = -243.200 \text{ MPa}, \quad |R| = SE\alpha\Delta T \approx 16.045 \text{ kN}. \quad (11)$$

This is not fully triaxial restraint and does not include local tube-sheet stress. The compressed member may buckle and the assumed material may yield; neither is assessed here. A converged linear answer does not establish structural safety.

5 Thermal FE load and proof

For a linear element of length ℓ , the actual assembled matrices are

$$\mathbf{k}_e = \frac{ES}{\ell} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad \mathbf{f}_{th,e} = ES\alpha\Delta T \begin{bmatrix} -1 \\ 1 \end{bmatrix}. \quad (12)$$

The script solves the free displacement degrees of freedom and recovers reactions from $\mathbf{R} = \mathbf{K}\mathbf{u} - \mathbf{f}_{th}$. Meshes of 2, 4, 8 and 16 elements reproduce the uniform analytical solution to floating-point precision. This problem lies exactly in the linear-element approximation space; lack of mesh error here does not prove mesh independence for a real exchanger.

6 What is and is not proved

Equation consistency, dimensions, pressure boundary tractions, end force balance, thermal boundary conditions, reactions, and idealized FE refinement are tested. There is no experiment, material certificate, exchanger assembly model, weld/contact model, fatigue assessment or hardware-release disposition. An LLM explanation is not an independent calculation. The October 12 local 3090 smoke test executed, but its thrust arithmetic failed; this notebook uses deterministic calculations instead.

7 Reproduction and references

Run `python verify.py` with Python and NumPy beside `inputs.json`. Compile this source using an existing TeX runtime. The website retains sources, calculated results and an SHA-256 manifest; hashes establish custody, not correctness.

- NPTEL, *Boundary Value Problem: Formulation*, Chapter 7, thick-cylinder elasticity: [primary course reference](#). Indexed source; full page retrieval timed out during this publication.
- COMSOL, *How to Provide Structural Stability in Thermal Expansion Simulations* (2018), restraint and rigid-motion cautions: [primary software-developer explanation](#).

Revision history. Notebook Rev 1.0, October 12, 2026: new synthetic public proof example. This does not supersede or rescore any historical Aero qualification or software revision.